Derivation of Eq. (A.13) in HPW paper [B. D. Hughes, B. A. Pailthorpe, and L. R. White, J. Fluid Mech. **110**, 349 (1981)].

In HPW paper Eq. (A. 13) appears as a result of the Mellin transform of Eq. (A. 9) followed by the contour integration:

$$T(\epsilon) = \int_0^\infty du \frac{j_0(u) J_0(u)}{u+\epsilon} \qquad \text{(A 9)}$$

$$\mathscr{M}\{T(\epsilon); \epsilon \to s\} = \frac{\pi^{\frac{1}{2}} 2^s \Gamma(\frac{1}{2}s)\Gamma(\frac{3}{2}-s)}{4 \sin \pi s \, \Gamma^2(\frac{3}{2}-\frac{1}{2}s)\Gamma(1-\frac{1}{2}s)} \quad (0 < \operatorname{Re} s < 1). \qquad \text{(A 10)}$$

$$T^{(2)}(\epsilon) = \frac{\pi^{\frac{1}{2}}}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}\epsilon)^{2n}\Gamma(2n+\frac{3}{2})}{\Gamma^2(n+1)\Gamma^2(n+\frac{3}{2})}\{\ln(2\epsilon) - \psi(2n+\tfrac{3}{2}) + \psi(n+1) + \psi(n+\tfrac{3}{2})\}. \qquad \text{(A 13)}$$

$\hookrightarrow \ln\left(\frac{2}{\varepsilon}\right)$

$$\frac{1}{2\pi i}\frac{\pi^{\frac{3}{2}}}{4}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{2}{\varepsilon}\right)^s \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{3}{2}-s\right)}{\sin(\pi s)\,\Gamma^2\left(\frac{3}{2}-\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)} \qquad (*)$$

$c = 0 < Re(s) < 1$

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\dots(z+n)}$$ $\Rightarrow$ poles for negative values of $z = 0, -1, -2, \dots$

In our case

$$\Gamma\left(\frac{s}{2}\right) = \frac{\Gamma\left(\frac{s}{2}+n+1\right)}{\left(\frac{s}{2}\right)\left(\frac{s}{2}+1\right)\dots\left(\frac{s}{2}+n\right)}$$ $\Rightarrow$ poles for $s = -2k$, $k = 0, 1, 2, \dots$

$\sin(\pi s) = 0$ for $\cancel{\pi} s = -\cancel{\pi} k$, $k = 0, 1, 2, \dots$

$s = -k$

Let's consider negative even values of $s = 0, -2, -4, -6$.
The integrand in $(*)$ has a pole of the second order at $s = -2k$, $k = 0, 1, 2, \dots$

Let's denote the integrand as

$$\frac{g(s)}{h(s)}$$ with $h(s) = \left(\frac{s}{2}+n\right)\sin(\pi s)$

and $$g(s) = \left(\frac{2}{\varepsilon}\right)^s \frac{\Gamma\left(\frac{s}{2}+n+1\right)}{\left(\frac{s}{2}\right)\left(\frac{s}{2}+1\right)\dots\left(\frac{s}{2}+n-1\right)} \times$$

$$\times \frac{\Gamma\left(\frac{3}{2}-s\right)}{\Gamma^2\left(\frac{3}{2}-\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}$$

Residue of the integrand at $z = z_0$ for the pole at z_0 of the second order is given by:

$$\text{Res}\Big|_{z=z_0} = 2\,\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\,\frac{g(z_0)\,h'''(z_0)}{[h''(z_0)]^2}$$

In our case $z_0 = -2n$, $n = 0, 1, 2, \ldots$

$$h''\Big|_{s=-2n} = \pi \qquad h'''\Big|_{s=-2n} = 0$$

$$\Rightarrow \text{Res}\Big|_{s=-2n} = \frac{2}{\pi}\, g'(z_0)$$

$$g' = \frac{d}{ds}\left\{\left(\frac{2}{\varepsilon}\right)^s \frac{\Gamma\left(\frac{s}{2}+n+1\right)\Gamma\left(\frac{3}{2}-s\right)}{\Gamma^2\left(\frac{3}{2}-\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\right\}\Bigg|_{s=-2n} \times$$

(2.1) $$\times \frac{1}{\left(\frac{s}{2}\right)\left(\frac{s}{2}+1\right)\cdots\left(\frac{s}{2}+n-1\right)}\Bigg|_{s=-2n} +$$

$$+\left(\frac{2}{\varepsilon}\right)^{-2n} \frac{\overbrace{\Gamma(1)}^{1}\Gamma\left(\frac{3}{2}+2n\right)}{\Gamma^2\left(\frac{3}{2}+n\right)\Gamma(1+n)} \cdot \frac{d}{ds}\left(\frac{1}{\left(\frac{s}{2}\right)\left(\frac{s}{2}+1\right)\cdots\left(\frac{s}{2}+n-1\right)}\right)\Bigg|_{s=-2n}$$

In Mathematica we find that:

$$\frac{d}{ds}\left\{\left(\frac{2}{\varepsilon}\right)^s \frac{\Gamma\left(\frac{s}{2}+n+1\right)\Gamma\left(\frac{3}{2}-s\right)}{\Gamma^2\left(\frac{3}{2}-\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\right\}\Bigg|_{s=-2n} =$$

$$= -\frac{2^{-1-2n}\left(\frac{1}{\varepsilon}\right)^{-2n}\Gamma\left(\frac{3}{2}+2n\right)}{\Gamma(1+n)\,\Gamma^2\left(\frac{3}{2}+n\right)}\left(\gamma - 2\log\left(\frac{2}{\varepsilon}\right) - \psi(1+n) - 2\psi\left(\frac{3}{2}+n\right) + 2\psi\left(\frac{3}{2}+2n\right)\right)$$

Also,

$$\left.\frac{1}{\left(\frac{s}{2}\right)\left(\frac{s}{2}+1\right)\cdots\left(\frac{s}{2}+n-1\right)}\right|_{s=-2n} = \frac{1}{(-n)(-n+1)\cdots(-n+n-1)} =$$

$$= (-1)^n \frac{1}{n(n-1)\cdots 1} = \frac{(-1)^n}{n!} = \frac{(-1)^n}{\Gamma(n+1)}$$

$$\left.\frac{d}{ds}\left(\frac{1}{\left(\frac{s}{2}\right)\left(\frac{s}{2}+1\right)\cdots\left(\frac{s}{2}+n-1\right)}\right)\right|_{s=-2n} =$$

$$= \left.\frac{d}{dz}\,\frac{1}{z(z+1)\cdots(z+n-1)}\right|_{z=-n} \cdot \frac{1}{2} =$$

$$= \frac{1}{2}\left(-\frac{1}{z^2}\,\frac{1}{(z+1)(z+2)\cdots(z+n-1)} - \frac{1}{z(z+1)^2(z+2)\cdots(z+n-1)} - \cdots\right)$$

$$= \frac{1}{2}(-1)\,\frac{1}{z(z+1)(z+2)\cdots(z+n-1)}\left.\left(\frac{1}{z}+\frac{1}{z+1}+\frac{1}{z+2}+\cdots\right)\right|_{z=-n}$$

$$= \left(\cancel{-}\frac{1}{2}\right)(-1)^n \underbrace{\frac{1}{n(n-1)(n-2)\cdots 1}}_{\Gamma(n+1)} \cancel{(-1)} \underbrace{\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots\frac{1}{1}\right)}_{\substack{\text{in Mathematica} \\ \text{was found to be} \\ \gamma+\psi(1+n)}} =$$

$$= \frac{1}{2}(-1)^n \frac{1}{\Gamma(n+1)}\left(\gamma+\psi(1+n)\right)$$

All together,

$$g'(z_0=-2n) = \frac{1}{2}\Bigg[\left(\frac{2}{\varepsilon}\right)^{-2n} \frac{\Gamma\left(\frac{3}{2}+2n\right)\ (-1)^n}{\Gamma^2(n+1)\,\Gamma^2\left(\frac{3}{2}+n\right)} \times$$

$$\times\Big(-\gamma + 2\log\left(\frac{2}{\varepsilon}\right) + \psi(1+n) + 2\psi\left(\frac{3}{2}+n\right)$$

$$-2\psi\left(\frac{3}{2}+2n\right)\Big) + (-1)^n \left(\frac{2}{\varepsilon}\right)^{-2n} \frac{\Gamma\left(\frac{3}{2}+2n\right)}{\Gamma^2(1+n)\,\Gamma^2\left(\frac{3}{2}+n\right)} \times$$

$$\times\left(\gamma + \psi(1+n)\right)\Bigg] =$$

$$= (-1)^n \left(\frac{\varepsilon}{2}\right)^{2n} \frac{\Gamma\left(\frac{3}{2}+2n\right)}{\Gamma^2(n+1)\,\Gamma^2\left(\frac{3}{2}+n\right)} \Big(\log\left(\frac{2}{\varepsilon}\right) + \psi(n+1)$$

$$+\psi\left(\frac{3}{2}+n\right) - \psi\left(\frac{3}{2}+2n\right)\Big)$$

$$\Rightarrow \sum_{z_0=-2n} \text{Res} = \frac{\pi^{\frac{1}{2}}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\varepsilon\right)^{2n} \Gamma\left(2n+\frac{3}{2}\right)}{\Gamma^2(n+1)\,\Gamma^2\left(\frac{3}{2}+n\right)} \times$$

$$\times\left\{\log\left(\frac{2}{\varepsilon}\right) - \psi\left(2n+\frac{3}{2}\right) + \psi(n+1) + \psi\left(n+\frac{3}{2}\right)\right\}.$$

This is eq. (A 13) in HPW.